Linear Functions

• System of linear equations

- Consider a function or mapping, $\mathbf{y} = \mathbf{A}\mathbf{x}$, i.e.,

$$\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ y_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ &\vdots \\ y_m &= a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{aligned}$$

where $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \in \mathbf{R}^m$, $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in \mathbf{R}^{m\times n}$, and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbf{R}^n$

(1) $\mathbf{y}$ is measurement or observation; $\mathbf{x}$ is unknown to be determined

(2) $\mathbf{x}$ is input; $\mathbf{y}$ is output

• Linear functions

- A function or mapping $f : \mathbf{R}^n \to \mathbf{R}^m$ is *linear* if

(1) $f(\mathbf{x}+\mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y}), \forall \mathbf{x}, \mathbf{y} \in \mathbf{R}^n$

(2) $f(\alpha\mathbf{x}) = \alpha f(\mathbf{x}), \forall \mathbf{x} \in \mathbf{R}^n \forall \alpha \in \mathbf{R}$

- Any linear function $f : \mathbf{R}^n \to \mathbf{R}^m$ can be written as $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$ for some $\mathbf{A} \in \mathbf{R}^{m\times n}$

• Interpretation of $y_i = \sum_{j=1}^{n} a_{ij}x_j$

(1) a_{ij} is a gain from *j*th input (x_j) to *i*th output (y_i)

(2) *i*th row of $\mathbf{A} \Leftrightarrow$ *i*th output

(3) *j*th column of $\mathbf{A} \Leftrightarrow$ *j*th input

(4) $a_{ij} = 0 \Leftrightarrow$ *i*th output (y_i) does not depend on *j*th input (x_j)

(5) a_{ip} dominates all a_{ij} for $j \neq p \Leftrightarrow y_i$ depends mainly on x_p

(6) a_{qj} dominates all a_{ij} for $i \neq q \Leftrightarrow x_q$ affects mainly y_i

(7) $\mathbf{A}$ is diagonal, i.e., $a_{ij} = 0$ for $i \neq j \Leftrightarrow y_i$ depends only on x_i

(8) Sparsity pattern of **A** determines input-output interactions

• **Interpretation of y = Ax**

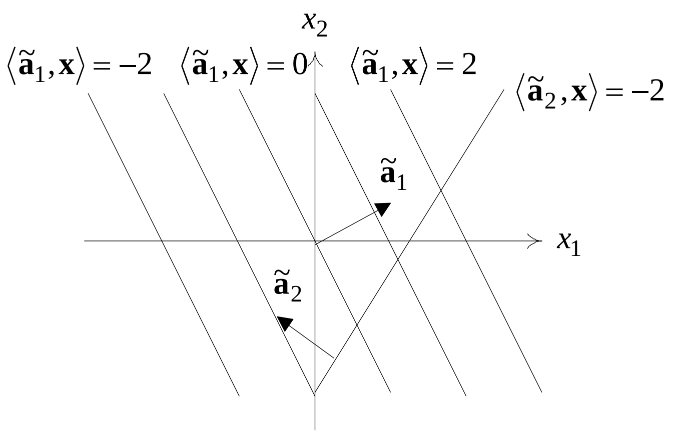
- Sum (linear combination) of columns. Write **A** as $\mathbf{A} = [\mathbf{a}_1 \mathbf{a}_2 \cdots \mathbf{a}_n]$ where $\mathbf{a}_j = [a_{1j}\, a_{2j} \cdots a_{mj}]^T \in \mathbf{R}^m$. Then,

$$\mathbf{y} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \sum_{j=1}^{n} x_j \mathbf{a}_j$$

- Inner product with rows. Write **A** as

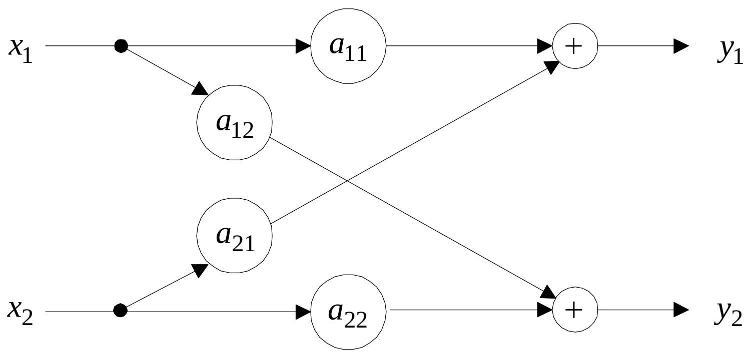
$$\mathbf{A} = \begin{bmatrix} \tilde{\mathbf{a}}_1^T \\ \tilde{\mathbf{a}}_2^T \\ \vdots \\ \tilde{\mathbf{a}}_m^T \end{bmatrix} \text{ where } \tilde{\mathbf{a}}_i = \begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{bmatrix} \in \mathbf{R}^n. \text{ Then } \mathbf{y} = \begin{bmatrix} \tilde{\mathbf{a}}_1^T\mathbf{x} \\ \tilde{\mathbf{a}}_2^T\mathbf{x} \\ \vdots \\ \tilde{\mathbf{a}}_m^T\mathbf{x} \end{bmatrix} \text{ or } y_i = \langle \tilde{\mathbf{a}}_i, \mathbf{x} \rangle.$$

- **x** is on intersection of all hyperplanes $\tilde{\mathbf{a}}_i^T\mathbf{x} = y_i$ for *i* = 1, 2, ..., *m*.



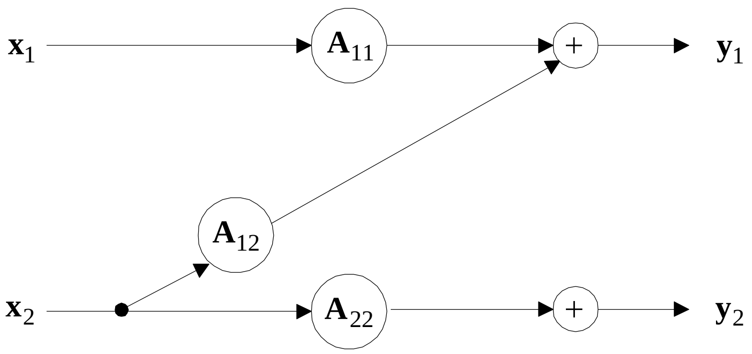
- Block diagram or signal flow graph

(1) For a system, $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$



(2) For a system, $\begin{bmatrix}\mathbf{y}_1\\ \mathbf{y}_2\end{bmatrix} = \begin{bmatrix}\mathbf{A}_{11} & \mathbf{A}_{12}\\ \mathbf{0} & \mathbf{A}_{22}\end{bmatrix}\begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\end{bmatrix}$, we can partition as $\mathbf{y}_1 = \mathbf{A}_{11}\mathbf{x}_1 + \mathbf{A}_{12}\mathbf{x}_2$

and $\mathbf{y}_2 = \mathbf{A}_{22}\mathbf{x}_2$

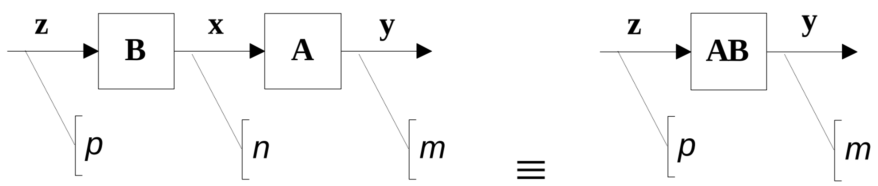


- **Composite functions or systems y = ABz**

- Matrix multiplication, $\mathbf{C} = \mathbf{AB} \in \mathbf{R}^{m \times p}$ where $\mathbf{A} \in \mathbf{R}^{m \times n}$ and $\mathbf{B} \in \mathbf{R}^{n \times p}$. Then,

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = \tilde{\mathbf{a}}_i^T \mathbf{b}_j = \langle \tilde{\mathbf{a}}_i, \mathbf{b}_j \rangle, \quad \mathbf{C} = \mathbf{AB} = \left[\mathbf{Ab}_1 \cdots \mathbf{Ab}_p\right], \text{ and } \mathbf{C} = \mathbf{AB} = \begin{bmatrix}\tilde{\mathbf{a}}_1^T\mathbf{B}\\ \vdots \\ \tilde{\mathbf{a}}_m^T\mathbf{B}\end{bmatrix}$$

- Composite interpretation.



- **Many engineering problems can be modelled as y = Ax**

- Estimation or inversion problems

(1) Model

(a) y_i is ith measurement or sensor reading

(b) x_j is jth parameter to be estimated

(c) a_{ij} is sensitivity of ith sensor to jth parameter

(2) Problems

(a) Given **y**, find **x**

(b) Find all **x**'s that can result in **y**

(c) If there is no **x** such that $\mathbf{y} = \mathbf{A}\mathbf{x}$, find the best $\hat{\mathbf{x}}$ such that $\mathbf{y} \approx \mathbf{A}\hat{\mathbf{x}}$ $\Rightarrow$ least square error solution and minimum norm solution

- Control or design problems

(1) Model

(a) **x** is design parameter or inputs or controls

(b) **y** is results or outputs or states

(c) **A** describes how input affects output

(2) Problems

(a) Given a desirable $\mathbf{y}^*$ (specifications), find **x** so that $\mathbf{y} = \mathbf{y}^*$

(b) Find all **x**'s that can result in $\mathbf{y} = \mathbf{y}^*$

(c) Among all **x**'s in (b), find a small or efficient one

- Signal processing problems

(1) Model

(a) y_i is output signal at time i

(b) x_j is input signal at time j

(2) Problems

(a) Given a desirable $\mathbf{y}^*$, find **A** so that $\mathbf{A}\mathbf{x} = \mathbf{y}^*$

(b) Given **A**, find **B** such that $\mathbf{B}\mathbf{y} = \mathbf{x}$ or $\mathbf{B}\mathbf{y} \approx \mathbf{x}$

Linear Algebra

• Vector space and subspace (see Vector Space)

- $V_1 = \mathbf{R}^n$, subspace of $\mathbf{R}^n$
- $V_2 = \{\mathbf{0}\}$ where $\mathbf{0} \in \mathbf{R}^n$, subspace of $\mathbf{R}^n$
- $V_3 = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$ where $\mathbf{v}_i \in \mathbf{R}^n$, subspace of $\mathbf{R}^n$
- $V_4 = \{\mathbf{x} : \mathbf{R}_+ \to \mathbf{R}^n \mid \mathbf{x} \text{ is differentiable}\}$ where $(\mathbf{x}+\mathbf{z})(t) = \mathbf{x}(t) + \mathbf{z}(t)$, $(\alpha\mathbf{x})(t) = \alpha\mathbf{x}(t)$ (a point in V_4 is a trajectory in $\mathbf{R}^n$)
- $V_5 = \{\mathbf{x} \in V_4 : \dot{\mathbf{x}} = \mathbf{A}\mathbf{x}\}$, subspace of V_4

• Linear independence, basis, dimension (see Vector Space)

- *Linearly independent* vs. *linearly dependent*
- $V =$ $\text{span}\{\mathbf{v}_i\}_{i=1}^n$ and $\{\mathbf{v}_i\}_{i=1}^n$ are linearly independent

  (a) $\{\mathbf{v}_i\}_{i=1}^n$ is a *basis* and *dimension* of V is n.

  (b) $\forall \mathbf{x} \in V, \mathbf{x} = \sum_{i=1}^{n} c_i \mathbf{v}_i$, $\{c_i\}_{i=1}^n$ is unique

  (c) Basis is not unique, i.e., V may have infinite number of bases

  (d) All bases have the same number of vectors (equal to dimension)

  (e) Any linearly independent set of vectors in V can be extended to a basis

  (f) Any spanning set of V can be reduced to a basis

• Real matrix and vector

- $\mathbf{R}^{m\times n}$ denotes the vector space of all $m\times n$ *real matrices*:

$$\mathbf{A} \in \mathbf{R}^{m\times n} \Leftrightarrow \mathbf{A} = \left[a_{ij}\right] = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}.$$

- $\mathbf{R}^m$ (i.e., $\mathbf{R}^{m\times 1}$) denotes the vector space of all $m\times 1$ *real column vectors*.

- For $\mathbf{x} \in \mathbf{R}^m$ and $\mathbf{y} \in \mathbf{R}^n$, $\mathbf{x}\mathbf{y}^T \in \mathbf{R}^{m \times n}$ is the *outer product*.
- For $\mathbf{x} \in \mathbf{R}^n$ and $\mathbf{y} \in \mathbf{R}^n$, $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y} \in \mathbf{R}^{1 \times 1}$ is the *inner product*.

• Complex matrices

- For $\mathbf{A} \in \mathbf{C}^{m \times n}$, then $\mathbf{A}^H = \left[\overline{a_{ji}} \right]$.
- For $\mathbf{x} \in \mathbf{C}^n$ and $\mathbf{y} \in \mathbf{C}^n$, $\mathbf{x}^H \mathbf{y} = \sum_{i=1}^{n} \overline{x_i} y_i = \overline{\mathbf{y}^H \mathbf{x}}$ is the *inner product*.
- $\mathbf{A} \in \mathbf{C}^{n \times n}$ is *unitary* if $\mathbf{A}^H \mathbf{A} = \mathbf{I}_n$, *Hermitian* if $\mathbf{A}^H = \mathbf{A}$, and *positive definite* if $\mathbf{x}^H \mathbf{A} \mathbf{x} > 0, \forall \mathbf{x} \in \mathbf{C}^n$ and $\mathbf{x} \neq \mathbf{0}$.

• Vector norm (Euclidean norm, see normed vector space)

- For $\mathbf{x} \in \mathbf{R}^n$, $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots x_n^2} = \sqrt{\mathbf{x}^T \mathbf{x}}$
- $\operatorname{dist}(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$

• Inner product (see Inner Product Space)

- For $\mathbf{x}, \mathbf{y} \in \mathbf{C}^n$, $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^H \mathbf{y} = x_1^* y_1 + \cdots x_n^* y_n = \sum_{i=1}^{n} x_i^* y_i$
- For $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$, $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y} = x_1 y_1 + \cdots x_n y_n = \sum_{i=1}^{n} x_i y_i$
- Row vector $\mathbf{x}^T$ represents a linear functional: $\mathbf{R}^n \to \mathbf{R}$
- $\left\{ \mathbf{x} : \mathbf{x}^T \mathbf{y} \leq 0 \right\}$ defines a *halfspace* with boundary passing $\mathbf{0}$ and outward normal vector $\mathbf{y}$

• Null space of a matrix

- For a matrix $\mathbf{A} \in \mathbf{R}^{m \times n}$, the *null space* of $\mathbf{A}$ is

$$N(\mathbf{A}) = \left\{ \mathbf{x} \in \mathbf{R}^n : \mathbf{A}\mathbf{x} = 0 \right\}.$$

(1) $N(\mathbf{A})$ is set of vectors mapped to zero by $\mathbf{y} = \mathbf{A}\mathbf{x}$

(2) $N(\mathbf{A})$ is set of vectors orthogonal to all rows of $\mathbf{A}$; $y_i = \langle \tilde{\mathbf{a}}_i, \mathbf{x} \rangle = 0$ for all i

- If $N(\mathbf{A}) = \{\mathbf{0}\}$,

(1) $\mathbf{y} = \mathbf{A}\mathbf{x}$ uniquely determines $\mathbf{x}$ (the linear transformation does not loose information, i.e., can be inversed)

(2) Since $\mathbf{y} = \sum_{i=1}^{n} x_i \mathbf{a}_i$ and $\{x_i\}_{i=1}^{n}$ is unique, $\{\mathbf{a}_i\}_{i=1}^{n}$ is a basis

(3) $\mathbf{A}$ has a left inverse, i.e., $\exists \mathbf{B} \in \mathbf{R}^{n \times m}$ s.t. $\mathbf{BA} = \mathbf{I}_n$

(4) $\det(\mathbf{A}^T\mathbf{A}) \neq 0$

- Meanings of $\mathbf{z} \in N(\mathbf{A})$

(1) Ambiguity in $\mathbf{x}$. Given $\mathbf{y} = \mathbf{A}\mathbf{x}$ and $\mathbf{z} \in N(\mathbf{A})$,

(a) $\mathbf{y} = \mathbf{A}(\mathbf{x} + \mathbf{z})$ $\Rightarrow \mathbf{z}$ is undetectable

(b) If $\mathbf{y} = \mathbf{A}\hat{\mathbf{x}}$, then $\hat{\mathbf{x}} = \mathbf{x} + \mathbf{z}$ $\Rightarrow \mathbf{x}$ and $\hat{\mathbf{x}}$ are indistinguishable

(2) Freedom of input choice. Given $\mathbf{y} = \mathbf{A}\mathbf{x}$ and $\mathbf{z} \in N(\mathbf{A})$

(a) $\mathbf{0} = \mathbf{A}\mathbf{z}$ $\Rightarrow \mathbf{z}$ is input with no result

(b) $\hat{\mathbf{x}} = \mathbf{x} + \mathbf{z}$ provides different input choice for the same result

• Range of a matrix

- For a matrix $\mathbf{A} \in \mathbf{R}^{m \times n}$, the <u>*range*</u> of $\mathbf{A}$ is

$$R(\mathbf{A}) = \{\mathbf{y} \in \mathbf{R}^m : \mathbf{y} = \mathbf{A}\mathbf{x} \text{ for some } \mathbf{x} \in \mathbf{R}^n\} = \text{span}\{\mathbf{a}_i\}_{i=1}^{n}$$

- $\mathbf{A}$ is called <u>*onto*</u> if $R(\mathbf{A}) = \mathbf{R}^m$

(1) $\text{span}\{\mathbf{a}_i\}_{i=1}^{n} = \mathbf{R}^m$ and $n \geq m$

(2) $\mathbf{y} = \mathbf{A}\mathbf{x}$ can be solved for $\mathbf{x}$

(3) $\mathbf{A}$ has a right inverse, i.e., $\exists \mathbf{B} \in \mathbf{R}^{n \times m}$ s.t. $\mathbf{AB} = \mathbf{I}_m$

(4) $\{\tilde{\mathbf{a}}_j\}_{j=1}^{m}$ are linearly independent

(5) $N(\mathbf{A}^T) = \{\mathbf{0}\}$

(6) $\det(\mathbf{A}\mathbf{A}^T) \neq 0$

- Meanings of $\mathbf{v} \in R(\mathbf{A})$ and $\mathbf{w} \notin R(\mathbf{A})$ given $\mathbf{y} = \mathbf{A}\mathbf{x}$

(1) $\mathbf{y}$ is a measurement of $\mathbf{x}$, $R(\mathbf{A})$ is the possible results

(a) $\mathbf{y} = \mathbf{v}$ is a possible or consistent sensor signal

(b) $\mathbf{y} = \mathbf{w}$ is an impossible or inconsistent sensor signal (sensor failure or wrong model)

(2) **y** is an output for input **x**, $R(\mathbf{A})$ is the achievable outputs

(a) **v** is a possible output

(b) **w** cannot be an output

• Rank of a matrix

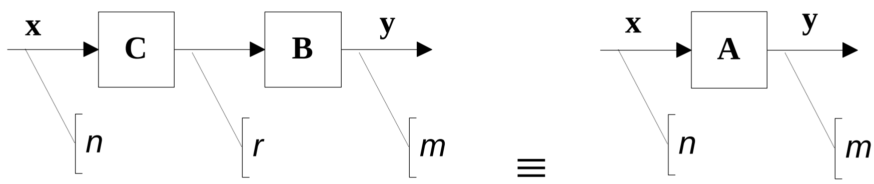
- For any matrix $\mathbf{A} \in \mathbf{R}^{m \times n}$, rank(**A**) = dim[$R$(**A**)]
- rank(**A**) is the maximal number of independent columns or rows ⇒

$$\text{rank}(\mathbf{A}) \le \min(m, n)$$

- $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}^T)$
- rank(**A**) + dim[N(**A**)] = n or dim[R(**A**)] + dim[N(**A**)] = n ⇒ each dimension of input **x** is either crushed to zero or ends up in nonzero output
- Rank of product; $\text{rank}(\mathbf{BC}) \le \min\{\text{rank}(\mathbf{B}),\ \text{rank}(\mathbf{C})\}$

(1) If **A** = **BC** with $\mathbf{A} \in \mathbf{R}^{m \times n}$, $\mathbf{B} \in \mathbf{R}^{m \times r}$, and $\mathbf{C} \in \mathbf{R}^{r \times n}$, then $\text{rank}(\mathbf{A}) \le r$

(2) If $\text{rank}(\mathbf{A}) = r$, then **A** = **BC** with $\mathbf{A} \in \mathbf{R}^{m \times n}$, $\mathbf{B} \in \mathbf{R}^{m \times r}$, and $\mathbf{C} \in \mathbf{R}^{r \times n}$



(a) $\text{rank}(\mathbf{A}) = r$ is the minimal size of vector needed to faithfully reconstruct **y** from **x**

(b) Two step computation, i.e., **z** = **Cx** and **y** = **Bz** needs $(m + n)r$ operations compared to mn oprtations in **y** = **Ax**

- **A** is *full rank* if $\text{rank}(\mathbf{A}) = \min\{m, n\}$

(1) If $m = n$, full rank means nonsingular

(2) If $m > n$ (narrow), full rank means columns are independent

(3) If $m < n$ (wide), full rank means rows are independent

• Inverse of a matrix

- A matrix $\mathbf{A} \in \mathbf{R}^{n \times n}$ is *invertible* or *nonsingular* if det(**A**) ≠ 0. Followings are equivalent to det(**A**) ≠ 0.

(1) $\{\mathbf{a}_i\}_{i=1}^n$ are a basis for $\mathbf{R}^n$

(2) $\{\tilde{\mathbf{a}}_j\}_{j=1}^n$ are a basis for $\mathbf{R}^n$

(3) $\mathbf{y} = \mathbf{Ax}$ has a unique solution $\mathbf{x}$ for every $\mathbf{y}$

(4) $\exists \mathbf{A}^{-1}$ s.t. $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_n$

(5) $N(\mathbf{A}) = \{\mathbf{0}\}$

(6) $R(\mathbf{A}) = \mathbf{R}^n$

(7) $\det(\mathbf{A}^T\mathbf{A}) = \det(\mathbf{AA}^T) \neq 0$

(8) $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}^T) = n$

• Coordinates

- Standard basis vectors in $\mathbf{R}^n$ are $\{\mathbf{e}_i\}_{i=1}^n$ where $\mathbf{e}_i = [0\,0 \cdots 0\,1\,0 \cdots 0]^T$

- $\mathbf{x} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n$ and $\{x_i\}_{i=1}^n$ are coordinates of $\mathbf{x}$ in the standard basis

- Change of coordinates

(1) If $\{\mathbf{t}_i\}_{i=1}^n$ are another basis for $\mathbf{R}^n$, $\mathbf{x} = \tilde{x}_1\mathbf{t}_1 + \cdots + \tilde{x}_n\mathbf{t}_n$

(2) Define $\mathbf{T} = [\mathbf{t}_1\mathbf{t}_2 \cdots \mathbf{t}_n]$, then $\mathbf{x} = \mathbf{T}\tilde{\mathbf{x}}$ with $\tilde{\mathbf{x}} = [\tilde{x}_1 \cdots \tilde{x}_n]^T$

(3) $\tilde{\mathbf{x}} = \mathbf{T}^{-1}\mathbf{x}$

- Consider a linear transform $\mathbf{y} = \mathbf{Ax}$ for $\mathbf{A} \in \mathbf{R}^{n\times n}$,

$$\mathbf{x} = \mathbf{T}\tilde{\mathbf{x}},\ \ \mathbf{y} = \mathbf{T}\tilde{\mathbf{y}},\ \text{and}\ \ \tilde{\mathbf{y}} = \left(\mathbf{T}^{-1}\mathbf{AT}\right)\tilde{\mathbf{x}}$$

(1) $\mathbf{A} \to \mathbf{T}^{-1}\mathbf{AT}$: similarity transformation

(2) Similarity transformation by $\mathbf{T}$ expresses $\mathbf{y}$ in coordinates $\mathbf{T} = [\mathbf{t}_1\mathbf{t}_2 \cdots \mathbf{t}_n]$

• Orthogonal vectors

- The set of vectors $\{\mathbf{u}_i\}_{i=1}^k$ in $\mathbf{R}^n$ are <u>*orthogonal*</u> if $\mathbf{u}_i^T\mathbf{u}_j = 0$ whenever $i \neq j$ and <u>*orthonormal*</u> if $\mathbf{u}_i^T\mathbf{u}_j = \delta_{ij}$.

- Let $\mathbf{U} = [\mathbf{u}_1 \cdots \mathbf{u}_k]$, then $\mathbf{U}^T\mathbf{U} = \mathbf{I}_k$ and $\text{span}\{\mathbf{u}_i\}_{i=1}^k = R(\mathbf{U})$
- Geometric properties

(1) If $\mathbf{w} = \mathbf{Uz}$, then $\|\mathbf{w}\| = \|\mathbf{Uz}\| = \|\mathbf{z}\|$. Mapping **U** is <u>*isometric*</u>.

(2) $\langle \mathbf{Uz}, \mathbf{Uw} \rangle = \langle \mathbf{z}, \mathbf{w} \rangle$ and hence $\angle(\mathbf{Uz}, \mathbf{Uw}) = \angle(\mathbf{z}, \mathbf{w})$

(3) If $\mathbf{w} = \mathbf{Uz}$ and $\tilde{\mathbf{w}} = \mathbf{U}\tilde{\mathbf{z}}$, $\langle \mathbf{w}, \tilde{\mathbf{w}} \rangle = \langle \mathbf{z}, \tilde{\mathbf{z}} \rangle$

(4) If $k = n$, the mapping **U** is either <u>*rotation*</u> or <u>*reflection.*</u>

• Orthonormal basis and expansion

- The set of vectors $\{\mathbf{u}_i\}_{i=1}^n$ in $\mathbf{R}^n$ is an <u>*orthonormal basis*</u> if $\mathbf{U} = [\mathbf{u}_1 \cdots \mathbf{u}_n]$ is square and orthogonal, i.e., $\mathbf{U}^T\mathbf{U} = \mathbf{I}_n$.
- Then, $\mathbf{U}^{-1} = \mathbf{U}^T$, $\mathbf{U}\mathbf{U}^T = \mathbf{I}_n$, and $\sum_{i=1}^n \mathbf{u}_i\mathbf{u}_i^T = \mathbf{I}_n$
- $\mathbf{x} = \mathbf{U}\mathbf{U}^T\mathbf{x} = \sum_{i=1}^n (\mathbf{u}_i^T\mathbf{x})\mathbf{u}_i = \sum_{i=1}^n \langle \mathbf{u}_i, \mathbf{x} \rangle \mathbf{u}_i$

(1) $\langle \mathbf{u}_i, \mathbf{x} \rangle = \mathbf{u}_i^T\mathbf{x}$ is the component (or projection) of **x** in the direction of $\mathbf{u}_i$

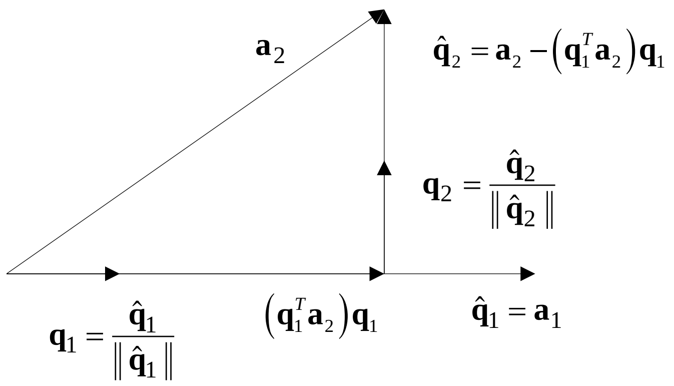
(2) $\mathbf{a} = \mathbf{U}^T\mathbf{x} = [\mathbf{u}_1^T\mathbf{x} \ \mathbf{u}_2^T\mathbf{x} \ \cdots \ \mathbf{u}_n^T\mathbf{x}]^T$ resolves **x** into $\mathbf{u}_i$ components

(3) **x** = **Ua** reconstructs **x** from its $\mathbf{u}_i$ components

(4) $\mathbf{x} = \mathbf{U}\mathbf{U}^T\mathbf{x} = \mathbf{Ua} = \sum_{i=1}^n a_i\mathbf{u}_i$ is the expansion of **x** in $\{\mathbf{u}_i\}_{i=1}^n$ basis

• Gram-Schmidt procedure

- Any independent vectors $\{\mathbf{a}_i\}_{i=1}^k$ in $\mathbf{R}^n$ can be transformed to orthonormal vectors $\{\mathbf{q}_i\}_{i=1}^k$ such that $\text{span}\{\mathbf{a}_i\}_{i=1}^r = \text{span}\{\mathbf{q}_i\}_{i=1}^r$ for $r \le k$.



- QR decomposition. Let $\mathbf{A} = [\mathbf{a}_1 \mathbf{a}_2 \cdots \mathbf{a}_k]$ and $\mathbf{Q} = [\mathbf{q}_1 \mathbf{q}_2 \cdots \mathbf{q}_k]$, then $\mathbf{A} = \mathbf{QR}$ with $\mathbf{Q}^T \mathbf{Q} = \mathbf{I}_k$ and $\mathbf{R}$ is upper triangular, invertible.

• Orthogonal complement

- A collection of subspaces $S_1, S_2, \cdots, S_p$ of $\mathbf{R}^n$ is *mutually orthogonal* if $\mathbf{x}^T \mathbf{y} = 0,\ \forall \mathbf{x} \in S_i$ and $\forall \mathbf{y} \in S_j$ for $i \neq j$.
- The *orthogonal complement* of a subspace $S \subset \mathbf{R}^n$ is

$$S^\perp = \{\mathbf{y} \in \mathbf{R}^n : \mathbf{y}^T \mathbf{x} = 0, \forall \mathbf{x} \in S\}.$$

- Note that $R(\mathbf{A})^\perp = N(\mathbf{A}^T)$.
- If $\{\mathbf{x}_i\}_{i=1}^p$ is an orthonormal basis for a subspace $S \subset \mathbf{R}^n$, then

$$\text{span}\{\mathbf{x}_i\}_{i=1}^p \oplus \text{span}\{\mathbf{x}_i\}_{i=p+1}^n = \mathbf{R}^n \quad \text{and} \quad S^\perp = \text{span}\{\mathbf{x}_i\}_{i=p+1}^n.$$

• Eigenvalues and eigenvectors

- For $\mathbf{A} \in \mathbf{R}^{n \times n}$, $\lambda \in \mathbf{C}$ is an eigenvalue of $\mathbf{A}$ if $\chi(\lambda) = \det(\lambda \mathbf{I} - \mathbf{A}) = 0$.
- For such $\lambda \in \mathbf{C}$, $\exists$ eigenvector, $\mathbf{v} \in \mathbf{C}^n, \mathbf{v} \neq \mathbf{0}$ suct that $\mathbf{Av} = \lambda \mathbf{v}$
- For such $\lambda \in \mathbf{C}$, $\exists$ left eigenvector, $\mathbf{w} \in \mathbf{C}^n, \mathbf{w} \neq \mathbf{0}$ suct that $\mathbf{w}^T \mathbf{A} = \lambda \mathbf{w}^T$
- Conjugate symmetry: $\mathbf{Av} = \lambda \mathbf{v} \Leftrightarrow \mathbf{A}\bar{\mathbf{v}} = \bar{\lambda}\bar{\mathbf{v}}$
- Interpretation: scaling by λ

(1) $\lambda \in \mathbf{R}, \lambda > 0$ $\Rightarrow$ **v** and **Av** are in the same direction
(2) $\lambda \in \mathbf{R}, \lambda < 0$ $\Rightarrow$ **v** and **Av** are in the opposite direction

(3) $\lambda \in \mathbf{R}, |\lambda| < 1$ $\Rightarrow$ **Av** is smaller than **v**

(4) $\lambda \in \mathbf{R}, |\lambda| > 1$ $\Rightarrow$ **Av** is larger than **v**

● Diagonalization

- Suppose $\{\mathbf{v}_i \in \mathbf{R}^{n\times 1}\}_{i=1}^{n}$ are linearly independent and $\mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}$ for $\mathbf{A} \in \mathbf{R}^{n\times n}$, then

  $\mathbf{AV} = \mathbf{V\Lambda}$ with $\mathbf{V} = [\mathbf{v}_1 \cdots \mathbf{v}_n]$ and $\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$

- Similarity transform by **V** diagonalize **A** since **V** is invertible $\Rightarrow$ $\mathbf{V}^{-1}\mathbf{AV} = \mathbf{\Lambda}$
- Conversely, $\mathbf{V}^{-1}\mathbf{AV} = \mathbf{\Lambda}$ $\Rightarrow$ $\{\mathbf{v}_i\}_{i=1}^{n}$ are linearly independent and $\mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}$
- **A** is *diagonalizable* if such **V** exists or **A** has linearly independent set of eigenvectors.
- **A** is *diagonalizable* if **A** has distinct eigenvalues.
- **A** is *defective* if it is not diagonalizable.
- Defective matrix can be put in *Jordan canonical form*.

● Symmetric matrix

- $\mathbf{A} \in \mathbf{R}^{n\times n}$ and $\mathbf{A} = \mathbf{A}^T$

  (1) Eigenvalues of **A** are real.

  (2) There is a set of orthonormal eigenvectors of **A**

    (a) $\exists \{\mathbf{q}_i\}_{i=1}^{n}$ such that $\mathbf{A}\mathbf{q}_i = \lambda_i \mathbf{q}_i$ and $\mathbf{q}_i^T \mathbf{q}_j = \delta_{ij}$

    (b) If $\{\lambda_i\}_{i=1}^{n}$ are distinct, corresponding eigenvectors are orthogonal. If not, choose eigenvectors so that they are orthogonal.

    (c) $\mathbf{Q}^{-1}\mathbf{AQ} = \mathbf{Q}^T\mathbf{AQ} = \mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$

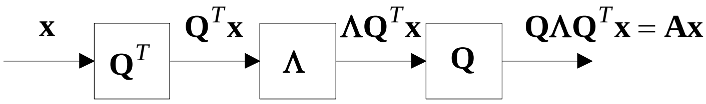
    (d) $\mathbf{A} = \mathbf{Q\Lambda Q}^T = \sum_{i=1}^{n} \lambda_i \mathbf{q}_i \mathbf{q}_i^T$: decomposition of linear combination of one-dimensional orthogonal projections

  (3) Interpretations of linear mapping $\mathbf{y} = \mathbf{Ax}$

    (a) Resolve **x** into $\mathbf{q}_i$ coordinates

(b) Scale coordinates by λ_i

(c) Reconstruct with basis $\mathbf{q}_i$



(4) Geometric interpretation

(a) Rotate by $\mathbf{Q}^T$

(b) Diagonal real scale by $\boldsymbol{\Lambda}$

(c) Rotate back by $\mathbf{Q}$

• Normal matrix

- $\mathbf{A} \in \mathbf{R}^{n\times n}$ is normal if $\mathbf{A}\mathbf{A}^T = \mathbf{A}^T\mathbf{A}$
  (1) Symmetric matrix is normal; $\mathbf{A} = \mathbf{A}^T$
  (2) Skew symmetric matrix is normal; $\mathbf{A} = -\mathbf{A}^T$
- $\mathbf{A}\mathbf{A}^T = \mathbf{A}^T\mathbf{A} \Leftrightarrow \exists\mathbf{Q}$ such that $\mathbf{Q}^T\mathbf{A}\mathbf{Q} = \boldsymbol{\Lambda}$, $\mathbf{A} = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^T = \sum_{i=1}^{n} \lambda_i \mathbf{q}_i \mathbf{q}_i^T$, $\mathbf{Q}\mathbf{Q}^T = \mathbf{I}_n$,

  $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$

• Gram matrix

- For $\mathbf{X} = [\mathbf{x}_1 \cdots \mathbf{x}_r] \in \mathbf{R}^{n\times r}$, Gram matrix or Grammian is $\mathbf{X}^T\mathbf{X} \in \mathbf{R}^{r\times r}$
- $\{\mathbf{x}_i\}_{i=1}^{r}$ are linearly independent $\Leftrightarrow$ $\det(\mathbf{X}^T\mathbf{X}) \neq 0$
- Factorization
  (1) QR factorization: $\mathbf{X} = \mathbf{Q}\mathbf{R}$, $\mathbf{Q} = \mathbf{X}\mathbf{R}^{-1}$, $\mathbf{Q}^T\mathbf{X} = \mathbf{R}$

  (2) Cholesky factorization: $\mathbf{G} = \mathbf{X}^T\mathbf{X} = \mathbf{R}^T\mathbf{Q}^T\mathbf{Q}\mathbf{R} = \mathbf{R}^T\mathbf{R}\,(= \mathbf{L}\mathbf{U} = \mathbf{L}\mathbf{D}\mathbf{U})$

• Quadratic forms

- A function $f: \mathbf{R}^n \to \mathbf{R}$ of the form $f(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x} = \sum_{i,j=1}^{n} a_{ij} x_i x_j$ is a quadratic form.
- Examples: $\|\mathbf{B}\mathbf{x}\|^2 = \mathbf{x}^T\mathbf{B}^T\mathbf{B}\mathbf{x}$, $\sum_{i=2}^{n} (x_{i+1} - x_i)^2$, $\|\mathbf{F}\mathbf{x}\|^2 - \|\mathbf{G}\mathbf{x}\|^2$
- Uniqueness: if $\mathbf{x}^T\mathbf{A}\mathbf{x} = \mathbf{x}^T\mathbf{B}\mathbf{x}$ for all $\mathbf{x} \in \mathbf{R}^n$ and $\mathbf{A} = \mathbf{A}^T, \mathbf{B} = \mathbf{B}^T$, then $\mathbf{A} = \mathbf{B}$.

- $\{\mathbf{x} : f(\mathbf{x}) = a\}$ is a quadratic surface.

- $\{\mathbf{x} : f(\mathbf{x}) \le a\}$ is a quadratic region.

- If $\mathbf{A} = \mathbf{A}^T$, $\mathbf{A} = \mathbf{Q}\Lambda\mathbf{Q}^T$ with $\lambda_1 \ge \cdots \ge \lambda_n$, then

  (1) $\lambda_n \mathbf{x}^T\mathbf{x} \le \mathbf{x}^T\mathbf{A}\mathbf{x} \le \lambda_1 \mathbf{x}^T\mathbf{x}$, $\lambda_n = \lambda_{\min}$ and $\lambda_1 = \lambda_{\max}$

  (2) $\mathbf{q}_1^T\mathbf{A}\mathbf{q}_1 = \lambda_1 \|\mathbf{q}_1\|^2$ and $\mathbf{q}_n^T\mathbf{A}\mathbf{q}_n = \lambda_n \|\mathbf{q}_n\|^2$

- For $\mathbf{A} = \mathbf{A}^T \in \mathbf{R}^n$,
  (1) If $\mathbf{x}^T\mathbf{A}\mathbf{x} \ge 0$ for all $\mathbf{x}$,
    (a) $\mathbf{A}$ is positive semidefinite and $\mathbf{A} \ge 0$
    (b) $\mathbf{A} \ge 0$ iff $\lambda_{\min}(\mathbf{A}) \ge 0$
  (2) If $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0$ for all $\mathbf{x}$,
    (a) A is positive definite and $\mathbf{A} > 0$
    (b) $\mathbf{A} > 0$ iff $\lambda_{\min}(\mathbf{A}) > 0$
  (3) $\mathbf{A}$ is negative semidefinite if $-\mathbf{A} \ge 0$
  (4) $\mathbf{A}$ is negative definite if $-\mathbf{A} > 0$
  (5) If $\mathbf{B} = \mathbf{B}^T$,
    (a) $\mathbf{A} \ge \mathbf{B}$ if $\mathbf{A} - \mathbf{B} \ge 0$
    (b) $\mathbf{A} < \mathbf{B}$ if $\mathbf{A} - \mathbf{B} < 0$
    (c) $\mathbf{A} > \mathbf{B}$ means $\mathbf{x}^T\mathbf{A}\mathbf{x} > \mathbf{x}^T\mathbf{B}\mathbf{x}$ for all $\mathbf{x} \ne \mathbf{0}$

- Ellipsoids: with $\mathbf{A} = \mathbf{A}^T$, the set $\{\mathbf{x} : \mathbf{x}^T\mathbf{A}\mathbf{x} \le 1\}$ is an ellipsoid in $\mathbf{R}^n$ centered at $\mathbf{0}$.

  (1) Semi-axes: $\mathbf{s}_i = \lambda_i^{-1/2}\mathbf{q}_i$

    (a) Eigenvectors define directions of semiaxes
    (b) Eigenvalues determine lengths of semiaxes

    (c) $(\mathbf{q}_1, \lambda_1 = \lambda_{\max})$ direction: smallest length, thin

    (d) $(\mathbf{q}_n, \lambda_n = \lambda_{\min})$ direction: largest length, fat

  (2) $\sqrt{\lambda_{\max}/\lambda_{\min}}$ = maximum eccentricity

• Matrix norms

- For $\mathbf{A} \in \mathbf{R}^{m \times n}$,

  (1) Matrix norm or spectral norm of **A**: $\|\mathbf{A}\| = \max_{\mathbf{x} \neq \mathbf{0}} \frac{\|\mathbf{A}\mathbf{x}\|}{\|\mathbf{x}\|}$ = maximum gain or amplification factor

  (2) Since $\max_{\mathbf{x} \neq \mathbf{0}} \frac{\|\mathbf{A}\mathbf{x}\|^2}{\|\mathbf{x}\|^2} = \max_{\mathbf{x} \neq \mathbf{0}} \frac{\mathbf{x}^T \mathbf{A}^T \mathbf{A}\mathbf{x}}{\|\mathbf{x}\|^2} = \lambda_{\max}\left(\mathbf{A}^T \mathbf{A}\right)$, $\|\mathbf{A}\| = \sqrt{\lambda_{\max}\left(\mathbf{A}^T \mathbf{A}\right)}$

  (3) Similarly, $\min_{\mathbf{x} \neq \mathbf{0}} \frac{\|\mathbf{A}\mathbf{x}\|}{\|\mathbf{x}\|} = \sqrt{\lambda_{\min}\left(\mathbf{A}^T \mathbf{A}\right)}$

  (4) Observations

  (a) $\mathbf{A}^T \mathbf{A} \in \mathbf{R}^{n \times n}$ is symmetric and $\mathbf{A}^T \mathbf{A} \geq 0 \;\Rightarrow\; \lambda_{\max}, \lambda_{\min} \geq 0$

  (b) $\left(\mathbf{q}_1, \lambda_{\max} = \lambda_1\left(\mathbf{A}^T \mathbf{A}\right)\right)$: maximum gain input direction

  (c) $\left(\mathbf{q}_n, \lambda_{\min} = \lambda_n\left(\mathbf{A}^T \mathbf{A}\right)\right)$: minimum gain input direction

- Properties of matrix norm

  (1) For $\mathbf{x} \in \mathbf{R}^{n \times 1}$, $\sqrt{\lambda_{\max}\left(\mathbf{x}^T \mathbf{x}\right)} = \sqrt{\mathbf{x}^T \mathbf{x}} = \|\mathbf{x}\|$

  (2) For any **x**, $\|\mathbf{A}\mathbf{x}\| \leq \|\mathbf{A}\| \, \|\mathbf{x}\|$

  (3) Scaling: $\|\alpha \mathbf{A}\| \leq |\alpha| \, \|\mathbf{A}\|$

  (4) Triangle inequality: $\|\mathbf{A} + \mathbf{B}\| \leq \|\mathbf{A}\| + \|\mathbf{B}\|$

  (5) $\|\mathbf{A}\| = 0 \Leftrightarrow \mathbf{A} = \mathbf{0}$

  (6) $\|\mathbf{A}\mathbf{B}\| \leq \|\mathbf{A}\| \|\mathbf{B}\|$

- For $\mathbf{A} \in \mathbf{C}^{m \times n}$,

  (1) F-norm (*Frobenius norm*) is $\|\mathbf{A}\|_F = \left[\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right]^{1/2}$

(2) <u>*p*-norm</u> is $\|\mathbf{A}\|_p = \sup_{\mathbf{x}\neq\mathbf{0}} \frac{\|\mathbf{A}\mathbf{x}\|_p}{\|\mathbf{x}\|_p}$.

• Matrix inversion formula

- Inverse of a partitioned matrix

$$\mathbf{R} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \Leftrightarrow \mathbf{R}^{-1} = \begin{bmatrix} \mathbf{E}^{-1} & \mathbf{F}\mathbf{H}^{-1} \\ \mathbf{H}^{-1}\mathbf{G} & \mathbf{H}^{-1} \end{bmatrix}$$

$$\mathbf{E} = \mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}$$
$$\mathbf{A}\mathbf{F} = -\mathbf{B}$$
$$\mathbf{G}\mathbf{A} = -\mathbf{C}$$
$$\mathbf{H} = \mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}$$

- Matrix inversion lemma

$$\mathbf{E} = \mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C} \Leftrightarrow \mathbf{E}^{-1} = \mathbf{A}^{-1} + \mathbf{F}\mathbf{H}^{-1}\mathbf{G}$$

- Partitioned matrix inverse

$$\mathbf{R} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \Leftrightarrow \mathbf{R}^{-1} = \begin{bmatrix} \mathbf{A}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{F} \\ \mathbf{I} \end{bmatrix} \mathbf{H}^{-1} \begin{bmatrix} \mathbf{G} & \mathbf{I} \end{bmatrix}$$

- Woodbury's identity (rank 1 update)

$$\mathbf{R} = \mathbf{R}_0 + \gamma^2 \mathbf{u}\mathbf{u}^T \Leftrightarrow \mathbf{R}^{-1} = \mathbf{R}_0^{-1} + \frac{\gamma^2}{1 + \gamma^2 \mathbf{u}^T \mathbf{R}_0^{-1} \mathbf{u}} \mathbf{R}_0^{-1} \mathbf{u}\mathbf{u}^T \mathbf{R}_0^{-1}$$

Singular Value Decomposition (SVD)

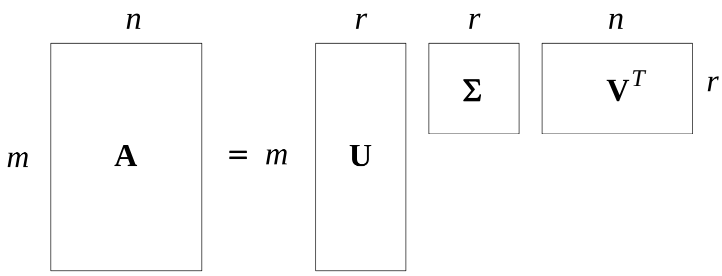
- **SVD of A**

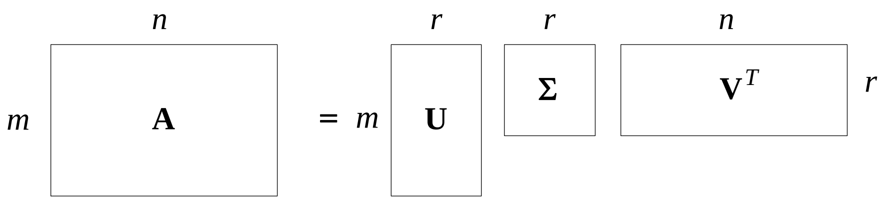
- $\mathbf{A} \in \mathbf{R}^{m \times n}$, $\text{rank}(\mathbf{A}) = r$
- $\mathbf{A} = \mathbf{U\Sigma V}^T = \sum_{i=1}^{r} \sigma_i \mathbf{u}_i \mathbf{v}_i^T$

(1) $\mathbf{U} = [\mathbf{u}_1 \cdots \mathbf{u}_r] \in \mathbf{R}^{m \times r}$, $\mathbf{U}^T\mathbf{U} = \mathbf{I}_r$, $\mathbf{u}_i$ are right or input singular vectors of **A**

(2) $\mathbf{V} = [\mathbf{v}_1 \cdots \mathbf{v}_r] \in \mathbf{R}^{m \times r}$, $\mathbf{V}^T\mathbf{V} = \mathbf{I}_r$, $\mathbf{v}_i$ are left or output singular vectors of **A**

(3) $\mathbf{\Sigma} = \text{diag}(\sigma_1, \cdots, \sigma_r)$ with $\sigma_1 \geq \cdots \geq \sigma_r > 0$, σ_i are nonzero singular values of **A**





- $\mathbf{A}^T\mathbf{A} = (\mathbf{U\Sigma V}^T)^T(\mathbf{U\Sigma V}^T) = \mathbf{V\Sigma}^2\mathbf{V}^T$

(1) $\mathbf{v}_i$ are eigenvectors of $\mathbf{A}^T\mathbf{A}$

(2) $\sigma_i = \sqrt{\lambda_i(\mathbf{A}^T\mathbf{A})}$ and $\lambda_i(\mathbf{A}^T\mathbf{A}) = 0$ for $i > r$

(3) $\|\mathbf{A}\| = \sqrt{\lambda_{\max}(\mathbf{A}^T\mathbf{A})} = \sigma_1$

- $\mathbf{AA}^T = (\mathbf{U\Sigma V}^T)(\mathbf{U\Sigma V}^T)^T = \mathbf{U\Sigma}^2\mathbf{U}^T$

(1) $\mathbf{u}_i$ are eigenvectors of $\mathbf{A}\mathbf{A}^T$

(2) $\sigma_i = \sqrt{\lambda_i\left(\mathbf{A}\mathbf{A}^T\right)}$ and $\lambda_i\left(\mathbf{A}\mathbf{A}^T\right) = 0$ for $i > r$

- $\{\mathbf{u}_i\}_{i=1}^{r}$ are orthonormal basis for $R(\mathbf{A})$
- $\{\mathbf{v}_i\}_{i=1}^{r}$ are orthonormal basis for $N(\mathbf{A})^{\perp}$

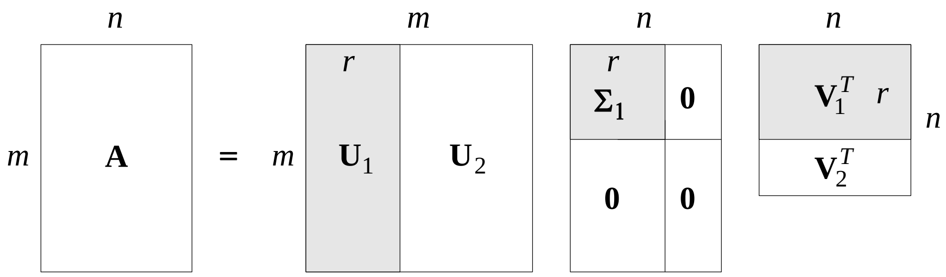
● **Full SVD**

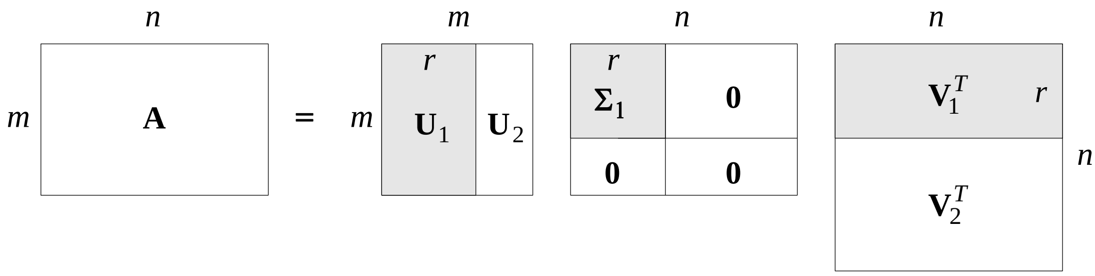
- $\mathbf{A} \in \mathbf{R}^{m\times n}$, $\text{rank}(\mathbf{A}) = r$
- $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T = \sum_{i=1}^{r} \sigma_i \mathbf{u}_i \mathbf{v}_i^T$

(1) $\mathbf{U}_1 = [\mathbf{u}_1 \cdots \mathbf{u}_r] \in \mathbf{R}^{m\times r}$, find $\mathbf{U}_2 = [\mathbf{u}_{r+1} \cdots \mathbf{u}_m] \in \mathbf{R}^{m\times(m-r)}$ such that $\mathbf{U} \in \mathbf{R}^{m\times m}$ and $\mathbf{U}^T\mathbf{U} = \mathbf{I}_m$

(2) $\mathbf{V}_1 = [\mathbf{v}_1 \cdots \mathbf{v}_r] \in \mathbf{R}^{n\times r}$, find $\mathbf{V}_2 = [\mathbf{v}_{r+1} \cdots \mathbf{v}_n] \in \mathbf{R}^{n\times(n-r)}$ such that $\mathbf{V} \in \mathbf{R}^{n\times n}$ and $\mathbf{V}^T\mathbf{V} = \mathbf{I}_n$

(3) $\mathbf{\Sigma}_1 = \text{diag}(\sigma_1, \cdots, \sigma_r)$ with $\sigma_1 \geq \cdots \geq \sigma_r > 0$, σ_i are nonzero singular values of $\mathbf{A}$, construct $\mathbf{\Sigma} = \begin{bmatrix} \mathbf{\Sigma}_1 & \mathbf{0}_{r\times(n-r)} \\ \mathbf{0}_{(m-r)\times r} & \mathbf{0}_{(m-r)\times(n-r)} \end{bmatrix} \in \mathbf{R}^{m\times n}$





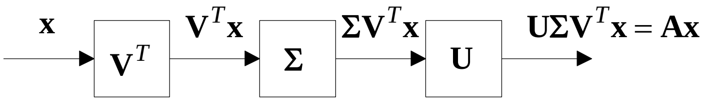
• Interpretations of SVD

- Decomposition of linear mapping $\mathbf{y} = \mathbf{A}\mathbf{x} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T\mathbf{x}$

  (1) $\mathbf{V}^T\mathbf{x}$: coefficients of $\mathbf{x}$ along input directions, $\{\mathbf{v}_i\}_{i=1}^{r}$

  (2) $\boldsymbol{\Sigma}\mathbf{V}^T\mathbf{x}$: scaling by $\{\sigma_i\}_{i=1}^{r}$

  (3) $\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T\mathbf{x}$: reconstruction along output directions, $\{\mathbf{u}_i\}_{i=1}^{r}$



- $\mathbf{A}\mathbf{v}_i = \sigma_i \mathbf{u}_i$

  (1) $\mathbf{v}_1$ is the most sensitive (highest gain) input direction

  (2) $\mathbf{u}_1$ is the most sensitive (highest gain) output direction

- Geometric interpretation of $\mathbf{y} = \mathbf{A}\mathbf{x} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T\mathbf{x}$

  (1) $\mathbf{V}^T$: rotation

  (2) $\boldsymbol{\Sigma}$: scaling; stretched or compressed or squashed to zero ($\sigma_i = 0$)

  (3) Zero padding (if $m > n$) or truncation (if $m < n$) to get m-vector

  (4) $\mathbf{U}$: rotation

• Pseudo-inverse

- For $\mathbf{A} \in \mathbf{R}^{m\times n}$, $\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T$ and $\mathbf{A}^+ = \mathbf{V}\boldsymbol{\Sigma}^{-1}\mathbf{U}^T$ in compact form

(1) If $m > n$ (narrow) and rank($\mathbf{A}$) = n, $\mathbf{A}^{+} = \mathbf{V}\mathbf{\Sigma}^{-1}\mathbf{U}^{T} = \left(\mathbf{A}^{T}\mathbf{A}\right)^{-1}\mathbf{A}^{T}$ $\Rightarrow$ least squares solution

(2) If $m < n$ (wide) and rank($\mathbf{A}$) = m, $\mathbf{A}^{+} = \mathbf{V}\mathbf{\Sigma}^{-1}\mathbf{U}^{T} = \mathbf{A}^{T}\left(\mathbf{A}\mathbf{A}^{T}\right)^{-1}$ $\Rightarrow$ minimum norm solution

- Properties

  (1) $\mathbf{A}^{+}\mathbf{A} = \mathbf{I}_n$

  (2) $\mathbf{A}\mathbf{A}^{+} = \mathbf{P}_A$ (see Projection)

  (3) $\mathbf{A}^{+}\mathbf{A}\mathbf{A} = \mathbf{A}$

  (4) $\mathbf{A}^{+}\mathbf{A}\mathbf{A}^{+} = \mathbf{A}^{+}$

• Projection

- Assume $R(\mathbf{A}) = \mathbf{R}^n$ for some matrix $\mathbf{A} = \left[\mathbf{a}_1 \cdots \mathbf{a}_p \mathbf{a}_{p+1} \cdots \mathbf{a}_n\right] \in \mathbf{R}^{m\times n}$, then

  (1) rank($\mathbf{A}$) = n

  (2) $\{\mathbf{a}_i\}_{i=1}^{n}$ are linearly independent.

  (3) $\text{span}\{\mathbf{a}_i\}_{i=1}^{n} = \mathbf{R}^n$

- Partition $\mathbf{A}$ into $\mathbf{S}$ and $\mathbf{N}$

  (1) $\mathbf{A} = [\mathbf{S} \quad \mathbf{N}]$

  (2) $\mathbf{S} = \left[\mathbf{a}_1 \cdots \mathbf{a}_p\right] \in \mathbf{R}^{m\times p}$ and let the subspace $S = \text{span}\{\mathbf{a}_i\}_{i=1}^{p} = R(\mathbf{S})$

  (3) $\mathbf{N} = \left[\mathbf{a}_{p+1} \cdots \mathbf{a}_n\right] \in \mathbf{R}^{m\times(n-p)}$ and let the subspace $N = \text{span}\{\mathbf{a}_i\}_{i=p+1}^{n} = R(\mathbf{N})$

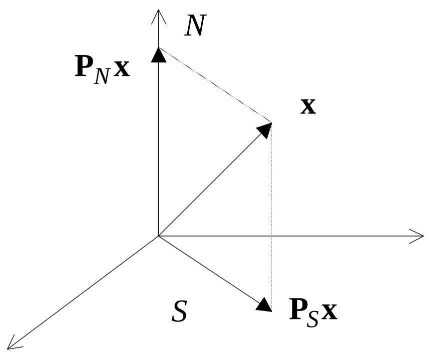
- By SVD, $\mathbf{S} = \mathbf{U}_S\mathbf{\Sigma}_S\mathbf{V}_S^{T}$ and $\mathbf{N} = \mathbf{U}_N\mathbf{\Sigma}_N\mathbf{V}_N^{T}$

  (1) $S = \text{span}\{\mathbf{a}_i\}_{i=1}^{p} = R(\mathbf{S}) = R(\mathbf{U}_S)$

  (2) $N = \text{span}\{\mathbf{a}_i\}_{i=p+1}^{n} = R(\mathbf{N}) = R(\mathbf{U}_N)$

  (3) $S \oplus N = \mathbf{R}^n$ and $S^{\perp} = N$

- Projection: for any $\mathbf{x} \in \mathbf{R}^n$, $\mathbf{x} = \mathbf{P}_S\mathbf{x} + \mathbf{P}_N\mathbf{x}$



(1) $\mathbf{P}_S = \mathbf{S}\left(\mathbf{S}^T\mathbf{S}\right)^{-1}\mathbf{S}^T = \mathbf{U}_S\mathbf{U}_S^T$ and $\mathbf{P}_N = \mathbf{N}\left(\mathbf{N}^T\mathbf{N}\right)^{-1}\mathbf{N}^T = \mathbf{U}_N\mathbf{U}_N^T = \mathbf{I}_n - \mathbf{U}_S\mathbf{U}_S^T$

(2) Idempotent: $\mathbf{P}_S^T = \mathbf{P}_S = \mathbf{P}_S^2$ and $\mathbf{P}_N^T = \mathbf{P}_N = \mathbf{P}_N^2$

(3) $\mathbf{P}_S\mathbf{P}_N = \mathbf{P}_N\mathbf{P}_S = \mathbf{0}$ and $\mathbf{P}_S + \mathbf{P}_N = \mathbf{I}_n$

(4) Let $\mathbf{U} = \left[\mathbf{U}_S \quad \mathbf{U}_N\right]$

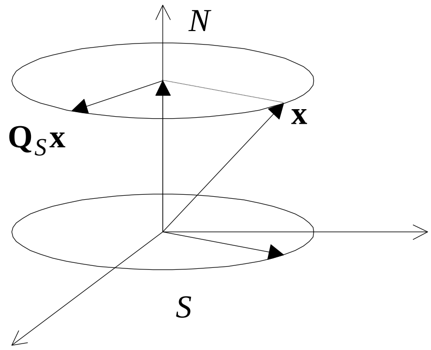
(a) $\mathbf{U}^T\mathbf{P}_S\mathbf{U} = \begin{bmatrix} \mathbf{I}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$

(b) $\mathbf{U}^T\mathbf{P}_N\mathbf{U} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{(n-p)} \end{bmatrix}$

(5) Eigenvalues of $\mathbf{P}_S$ and $\mathbf{P}_N$ are 1 and 0.

• **Rotation**

- Same setup as above in projection
- Rotate in the subspace *S* or rotate around the subspace *N*
- $\tilde{\mathbf{x}} = \mathbf{Q}_S\mathbf{x}$ with $\mathbf{Q}_S = \mathbf{U}_S\mathbf{Q}\mathbf{U}_S^T + \mathbf{P}_N$ and $\mathbf{Q}^T\mathbf{Q} = \mathbf{Q}\mathbf{Q}^T = \mathbf{I}_n$
- $\mathbf{P}_N\mathbf{x}$ component is not changed.
- $\mathbf{P}_S\mathbf{Q}_S = \mathbf{Q}_S\mathbf{P}_S$



• SVD in estimation and inversion

- Model: $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{v}$
  (1) $\mathbf{x}$ is what we want to estimate.
  (2) $\mathbf{y}$ is sensor measurements.
  (3) $\mathbf{v}$ is unknown noise or measurement error with $\|\mathbf{v}\| \le \alpha$

- Optimality of least squares estimator
  (1) Consider a linear estimator $\hat{\mathbf{x}} = \mathbf{B}\mathbf{y}$ with $\mathbf{BA} = \mathbf{I}$
  (2) Error is $\mathbf{e} = \hat{\mathbf{x}} - \mathbf{x} = \mathbf{B}\mathbf{v}$ and the set of possible estimation errors is an ellipsoid $\mathbf{e} \in E = \{\mathbf{B}\mathbf{v} : \|\mathbf{v}\| \le \alpha\}$
  (3) Uncertainty ellipsoid E must be small since $\|\mathbf{e}\| = \|\hat{\mathbf{x}} - \mathbf{x}\| \le \alpha \|\mathbf{B}\|$
  (4) Using SVD, we can prove that $\|\mathbf{A}^+\| = \|\mathbf{V}\mathbf{\Sigma}^{-1}\mathbf{U}^T\| = \left\|\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\right\| \le \|\mathbf{B}\|$

• Condition number

- Error analysis of $\mathbf{x} = \mathbf{A}^{-1}\mathbf{y}$ for $\mathbf{A} \in \mathbf{R}^{n\times n}$ and invertible
  (1) Assume error or noise in $\mathbf{y}$ as $\mathbf{y} + \delta\mathbf{y}$
  (2) $\mathbf{x} = \mathbf{x} + \delta\mathbf{x}$ with $\delta\mathbf{x} = \mathbf{A}^{-1}\delta\mathbf{y}$ and $\|\delta\mathbf{x}\| = \|\mathbf{A}^{-1}\delta\mathbf{y}\| \le \|\mathbf{A}^{-1}\|\,\|\delta\mathbf{y}\|$
  (3) $\dfrac{\|\delta\mathbf{x}\|}{\|\mathbf{x}\|} \le \|\mathbf{A}\|\|\mathbf{A}^{-1}\| \dfrac{\|\delta\mathbf{y}\|}{\|\mathbf{y}\|} = \kappa(\mathbf{A}) \dfrac{\|\delta\mathbf{y}\|}{\|\mathbf{y}\|}$
  (4) Condition number of $\mathbf{A}$ is $\kappa(\mathbf{A}) = \dfrac{\sigma_{\max}(\mathbf{A})}{\sigma_{\min}(\mathbf{A})}$ $\Rightarrow$ can compute using SVD
  (5) If $\kappa(\mathbf{A})$ is large, $\mathbf{A}$ is practically singular and ill-conditioned. If small, well-

conditioned.

(6) $\sigma_{\min}(\mathbf{A})$ is distance to nearest singular matrix.

- Use regularization for the solution of $\mathbf{y} = \mathbf{Ax}$ with ill-conditioned $\mathbf{A}$.

• Low rank approximation or model simplification

- $\mathbf{A} \in \mathbf{R}^{m\times n}$, $\text{rank}(\mathbf{A}) = r$, and $\mathbf{A} = \mathbf{U\Sigma V}^T = \sum_{i=1}^{r} \sigma_i \mathbf{u}_i \mathbf{v}_i^T$
- Assume $\sigma_1 > \sigma_2 > \cdots > \sigma_p$ and $\{\sigma_i\}_{i=p+1}^{r}$ are all small, then we can approximate **A** by the optimal rank *p* approximator,

$$\hat{\mathbf{A}} = \sum_{i=1}^{p} \sigma_i \mathbf{u}_i \mathbf{v}_i^T \quad \text{and} \quad \left\|\mathbf{A} - \hat{\mathbf{A}}\right\| = \left\|\sum_{i=p+1}^{r} \sigma_i \mathbf{u}_i \mathbf{v}_i^T\right\| = \sigma_{p+1} + \cdots + \sigma_r$$

Solution of Ax = b for A ∈ $\mathbf{R}^{n\times n}$

• Matrix multiplication, C = AB

- $c_{ij} = \tilde{\mathbf{a}}_i^T \mathbf{b}_j$, $\mathbf{c}_j = \mathbf{A}\mathbf{b}_j$, $\tilde{\mathbf{c}}_i = \tilde{\mathbf{a}}_i^T \mathbf{B}$
- $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$
- $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$ and $(\mathbf{B}+\mathbf{C})\mathbf{D} = \mathbf{BD}+\mathbf{CD}$
- $\mathbf{FE} \neq \mathbf{EF}$, in general

• Matrix transpose, $\mathbf{A}^T$

- $\left(\mathbf{A}^T\right)_{ij} = (\mathbf{A})_{ji}$
- $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$

• System of linear equations

- Consider a system of linear equations, $\mathbf{y} = \mathbf{Ax}$, i.e.,

$$\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ y_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ &\vdots \\ y_n &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{aligned}$$

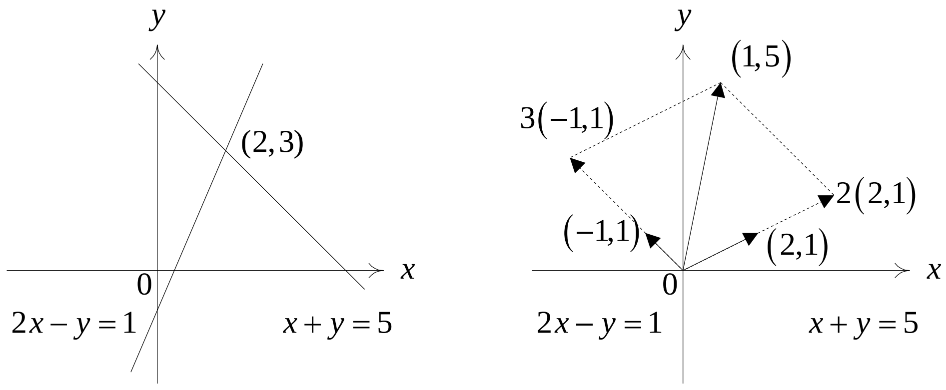
where $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \in \mathbf{R}^n$, $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \in \mathbf{R}^{n\times n}$, and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbf{R}^n$

- Three possibilities: one solution, no solution, infinite solutions

• Geometry

- Example:

$$\begin{aligned} 2x - y &= 1 \\ x + y &= 5 \end{aligned} \quad \text{or} \quad x\begin{bmatrix} 2 \\ 1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$



- In $\mathbf{R}^3$, $ax+by+cz=d$ is a plane. If $d = 0$, it passes the origin. If $d \neq 0$, it is parallel to the plane with $d = 0$. All of these planes are perpendicular to the vector $[a, b, c]^T$.

- $\mathbf{Ax}=\mathbf{b}$, $\mathbf{A} \in \mathbf{R}^{n\times n}$

  (1) Intersection of n planes:

$$\begin{bmatrix} \tilde{\mathbf{a}}_1^T \\ \tilde{\mathbf{a}}_2^T \\ \vdots \\ \tilde{\mathbf{a}}n_1^T \end{bmatrix} \mathbf{x} = \mathbf{b} \quad \text{or} \quad \begin{matrix} \tilde{\mathbf{a}}_1^T \mathbf{x} = b_1 \\ \tilde{\mathbf{a}}_2^T \mathbf{x} = b_2 \\ \vdots \\ \tilde{\mathbf{a}}_n^T \mathbf{x} = b_n \end{matrix}$$

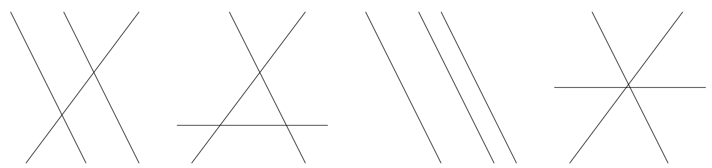
  (2) Linear combination of column vectors equals to **b**:

$$[\mathbf{a}_1 \mathbf{a}_2 \cdots \mathbf{a}_n]\mathbf{x} = \mathbf{b} \quad \text{or} \quad x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

  (3) Coordinates of intersection point = coefficients of linear combination

- <u>*Singularity*</u>: no solution or infinite solutions

  (1) No intersection point or infinite intersection points



(a) (b) (c) (d)

  (a) 2 parallel: no solution
  (b) No intersection: no solution
  (c) All parallel: no solution
  (d) Line of intersection: infinite solutions

  (2) $\mathbf{b} \notin \text{span}\{\mathbf{a}_i\}_{i=1}^n$

• Gaussian elimination

- (Forward elimination + backsubstitution) or (factorization + forward substitution + backsubstitution)
- Example:

$$\begin{Bmatrix} 2x+y+z=5 \\ 4x-6y=2 \\ -2x+7y+2z=9 \end{Bmatrix} \Rightarrow$$

$$\begin{bmatrix} \underline{2} & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{bmatrix} \Rightarrow \begin{bmatrix} \underline{2} & 1 & 1 & 5 \\ 0 & \underline{-8} & -2 & -12 \\ 0 & 8 & 3 & 14 \end{bmatrix} \Rightarrow \begin{bmatrix} \underline{2} & 1 & 1 & 5 \\ 0 & \underline{-8} & -2 & -12 \\ 0 & 0 & \underline{1} & 2 \end{bmatrix}$$

(1) Pivots cannot be zero.
(2) If pivot is zero, change order (rows) if possible (permutation).
(3) If all possible pivots are zero, **A** is singular (i.e., **A** is not invertible).

- After elimination with $O(n^3)$, solve for **x** by back substitution with $O(n^2)$.

• LU or LDU factorization

- For repeated solutions with different RHS, store multipliers during elimination process in a lower triangular matrix **L**. Main diagonal elements of **L** are all one.
- Store pivots and the rest of each row in an upper triangular matrix **U**. Main diagonal elements of **U** are pivots.

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \text{ and } \mathbf{U} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

- We can split **U** into **D** and **U** as

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \ \mathbf{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \mathbf{U} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

- Using LU factorization, **A** = **LU**. For a given **b**,
  (a) Forward substitution with $O(n^2)$: solve **Lc** = **b** for **c**
  (b) Backsubstitution with $O(n^2)$: solve **Ux** = **c** for **x**
- Using LDU factorization, **A** = **LDU**. For a given **b**,

(a) Forward substitution with $O(n^2)$: solve $\mathbf{Lc} = \mathbf{b}$ for $\mathbf{c}$
(b) Simple divisions with $O(n)$: solve $\mathbf{De} = \mathbf{c}$ for $\mathbf{e}$
(c) Backsubstitution with $O(n^2)$: solve $\mathbf{Ux} = \mathbf{e}$ for $\mathbf{x}$

• Inverse matrix

- $\mathbf{Ax} = \mathbf{b}$ has a unique solution. ⇔ $\mathbf{A}$ is nonsingular. ⇔ $\mathbf{A}^{-1}$ exists.
- If $\mathbf{A}^{-1}$ exists, $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ and $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ and $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$.
- All pivots are nonzero. ⇔ $\mathbf{A}^{-1}$ exists.
- All eigenvalues are nonzero. ⇔ $\mathbf{A}^{-1}$ exists.
- Determinant of $\mathbf{A}$ is nonzero. ⇔ $\mathbf{A}^{-1}$ exists.
- $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$, $(\mathbf{A}^{-1})^T = (\mathbf{A}^T)^{-1}$

• Symmetric nonsingular matrix

- $\mathbf{A}$ is symmetric ⇔ $\mathbf{A} = \mathbf{A}^T$
- If $\mathbf{A}$ is symmetric and nonsingular, $\mathbf{A} = \mathbf{LU} = \mathbf{LL}^T = \mathbf{U}^T\mathbf{U}$: *Cholesky factorization*
- If $\mathbf{A}$ is symmetric and nonsingular, $\mathbf{A} = \mathbf{LDU} = \mathbf{LDL}^T = \mathbf{U}^T\mathbf{DU}$

• Comments

- If all pivots are povitive, $\mathbf{A}$ is called *positive definite*.
- If $\mathbf{A}$ is *ill-conditioned*,
  (1) Computation is sensitive to a small numerical error.
  (2) Use *partial pivoting* where we pick up the largest pivot among all possible pivots. This requires permutation.
- If $\mathbf{A}$ is not ill-conditioned, it is well-conditioned.

Solution of Ax = b for A ∈ $\mathbf{R}^{m \times n}$

• Ax = b

- Scalar example: $ax = b$
  (a) $a \neq 0 \Rightarrow$ unique solution, nonsingular
  (b) $a = 0$ and $b = 0 \Rightarrow$ infinite number of solutions, underdetermined
  (c) $a = 0$ and $b \neq 0 \Rightarrow$ no solution, inconsistent
- $\mathbf{Ax} = \mathbf{b}$, $\mathbf{A} \in \mathbf{R}^{m\times n}$
  (a) Infinite number of solutions for every **b**
  (b) Infinite number of solutions for some **b** and no solution for other **b**
  (c) One solution for some **b** and no solution for other **b**
- Example

$$\mathbf{A} = \begin{bmatrix} \underline{1} & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \underline{1} & 3 & 3 & 2 \\ 0 & \underline{0} & \underline{3} & 1 \\ 0 & 0 & 6 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} \underline{1} & 3 & 3 & 2 \\ 0 & \underline{0} & \underline{3} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{U}$$

$$\mathbf{U} = \begin{bmatrix} \underline{1} & 3 & 3 & 2 \\ 0 & \underline{0} & \underline{3} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ : echelon form, } \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}, \text{ and } \mathbf{A} = \mathbf{LU}$$

- $\mathbf{PA} = \mathbf{LU}$ for any $\mathbf{A} \in \mathbf{R}^{m\times n}$ with permutation **P**

• Homogeneous solution, Ax = 0

- $\mathbf{Ax} = \mathbf{0} \Rightarrow \mathbf{Ux} = \mathbf{0}$
- Example

$$\begin{bmatrix} \underline{1} & 3 & 3 & 2 \\ 0 & 0 & \underline{3} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left\{ \begin{matrix} x_3 = -\frac{1}{3} x_4 \\ x_1 = -3x_2 - x_4 \end{matrix} \right\} \Rightarrow \mathbf{x} = \begin{bmatrix} -3x_2 - x_4 \\ x_2 \\ -\frac{1}{3} x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

- Two kinds of variables
  (a) Column with pivot $\Rightarrow$ basic variable (x_1 and x_3)
  (b) Column without pivot $\Rightarrow$ free variable (x_2 and x_4)
- Procedure for homogeneous solution

(a) Identify basic and free variables from $\mathbf{Ux} = \mathbf{0}$

(b) For each free variable, set it to 1 and set all other free variables to 0. Solve $\mathbf{Ux} = \mathbf{0}$

(c) Combinations of all solutions from (b) equal to $N(\mathbf{A})$

- Comments

(a) If $n > m$ (wide matrix), nontrivial homogeneous solutions ($\mathbf{x} \neq \mathbf{0}$) of $\mathbf{Ax} = \mathbf{0}$ exist.

(b) Number of free variables = $\dim\{N(\mathbf{A})\}$

(c) Finding homogeneous solutions is equivalent to finding $N(\mathbf{A})$.

• Inhomogeneous or particular solution, $\mathbf{Ax} = \mathbf{b} \neq \mathbf{0}$

- $\mathbf{Ax} = \mathbf{b} \Rightarrow \mathbf{Ux} = \mathbf{c}$
- Example

$$\begin{bmatrix} \underline{1} & 3 & 3 & 2 \\ 0 & 0 & \underline{3} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 - 2b_2 + 5b_1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix} = [\mathbf{a}_1 \mathbf{a}_2 \mathbf{a}_3 \mathbf{a}_4]$$

$$R(\mathbf{A}) = \text{span}\{\mathbf{a}_1, \mathbf{a}_3\} \quad \text{or} \quad R(\mathbf{A}) = \{(b_1, b_2, b_3) : b_3 - 2b_2 + 5b_1 = 0\} \quad : \text{plane}$$

If $\mathbf{b} = [1 \quad 5 \quad 5]^T$, $\begin{bmatrix} \underline{1} & 3 & 3 & 2 \\ 0 & 0 & \underline{3} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ and $\left\{ \begin{matrix} x_3 = 1 - \frac{1}{3}x_4 \\ x_1 = -2 - 3x_2 - x_4 \end{matrix} \right\}$.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\text{particular solution}} \underbrace{+ x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -\frac{1}{3} \\ 1 \end{bmatrix}}_{\text{homogeneous solution}}$$

- Procedure for particular solution

(a) $\mathbf{Ax} = \mathbf{b} \Rightarrow \mathbf{Ux} = \mathbf{c}$

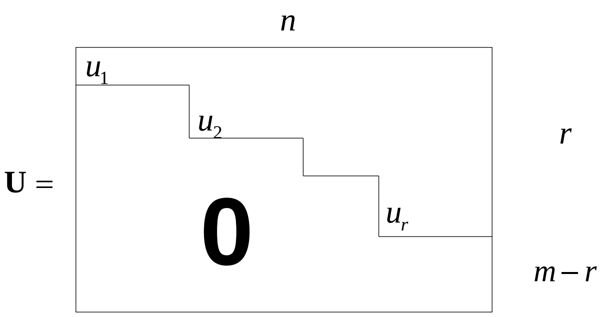
(b) Set all free variables to 0 and solve **U**$\mathbf{x}$ = **c** for **x**

• Total solution, Ax = b ≠ 0

- Total solution = particular solution + homogeneous solution ($\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$)
- Number of basic variables = r = number of pivots = rank(**A**) = number of independent columns = number of independent rows
- Number of free variables = $n - r$

• Comments

- Echelon form with r pivots, $\{u_i\}_{i=1}^{r}$



- $r = n \Rightarrow$ no free variable, $N(\mathbf{A}) = \{\mathbf{0}\}$, nontrivial homogeneous solution does not exist, $\{\mathbf{a}_i\}_{i=1}^{n}$ are linearly independent
- $r < n \Rightarrow (n - r)$ free variables, nontrivial homogeneous solution exists
- $r = m \Rightarrow$ particular solution exists for any $\mathbf{b} \in \mathbf{R}^m$, $R(\mathbf{A}) = \mathbf{R}^m$
- $r < m$

  (a) $c_{m-r} = c_{m-r+1} = \cdots = c_m = 0 \Rightarrow$ system is consistent and particular solution exists

  (b) $c_i \neq 0$ for some $m - r \leq i \leq m \Rightarrow$ system is inconsistent and particular solution does not exists
- rank(**A**) = $r \leq \min(m, n)$

• Inverse of $\mathbf{A} \in \mathbf{R}^{m \times n}$

- Left inverse: $\mathbf{BA} = \mathbf{I}_n, \mathbf{B} \in \mathbf{R}^{n \times m}$

- Right inverse: $\mathbf{AC} = \mathbf{I}_m, \mathbf{C} \in \mathbf{R}^{m \times n}$
- If both **B** and **C** exist, $\mathbf{B} = \mathbf{B}(\mathbf{AC}) = (\mathbf{BA})\mathbf{C} = \mathbf{C} = \mathbf{A}^{-1}$ and $m = n$

• Existence and uniqueness of solution of Ax = b, $\mathbf{A} \in \mathbf{R}^{m \times n}$

- Existence ($m \geq n$, narrow matrix): one or many solutions
  (a) **Ax = b** has "at least" one solution for every **b** iff $R(\mathbf{A}) = \mathbf{R}^m$.
  (b) In this case, $r = m$ and right inverse $\mathbf{C} \in \mathbf{R}^{m \times n}$ exists such that $\mathbf{AC} = \mathbf{I}_m$.
  (c) $\mathbf{B} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$ and $\text{rank}(\mathbf{A}^T\mathbf{A}) = n$
  (d) Least squares (LS) solution
- Uniqueness ($m \leq n$, wide matrix): one or zero solution
  (a) **Ax = b** has "at most" one solution for every **b** iff $n = r = \dim R(\mathbf{A})$.
  (b) In this case, $r = n$ and left inverse $\mathbf{B} \in \mathbf{R}^{n \times m}$ exists such that $\mathbf{BA} = \mathbf{I}_n$.
  (c) $\mathbf{C} = \mathbf{A}^T(\mathbf{AA}^T)^{-1}$ and $\text{rank}(\mathbf{AA}^T) = m$
  (d) Minimum norm solution
-

• Square matrix (*m* = *n*)

- $\mathbf{B} = \mathbf{C} = \mathbf{A}^{-1}$ iff $m = n = r$
- Followings are equivalent
  (a) $R(\mathbf{A}) = \mathbf{R}^m$, **Ax** = **b** has at least one solution for any **b**
  (b) $N(\mathbf{A}) = \{\mathbf{0}\}$, columns of **A** are linearly independent
  (c) $R(\mathbf{A}^T) = \mathbf{R}^n$
  (d) Rows of **A** are linearly independent
  (e) **PA** = **LDU** with all $d_{ii} \neq 0$ (nonzero pivot)
  (f) $\mathbf{A}^{-1}$ exists such that $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_n$
  (g) $\det(\mathbf{A}) \neq 0$
  (h) All eigenvalues of **A** are nonzero
  (i) $\mathbf{A}^T\mathbf{A}$ is positive definite
- Example: polynomial of degree (n – 1) has (n – 1) roots ⇒ Vandermonde's matrix

$$\mathbf{Ax} = \begin{bmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^{n-1} \\ 1 & t_2 & t_2^2 & \cdots & t_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & t_n & t_n^2 & \cdots & t_n^{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \mathbf{b}\text{ , }\mathbf{A}\text{ is nonsingular}$$

Four Subspaces of $\mathbf{A} \in \mathbf{R}^{m \times n}$

• **Four subspaces of $\mathbf{A} \in \mathbf{R}^{m \times n}$**

- Column space of **A** = range of **A** = $R(\mathbf{A}) = \text{span}\{\mathbf{a}_i\}_{i=1}^{n} \subset \mathbf{R}^m$, $\dim R(\mathbf{A}) = r$
- Nullspace of **A** = $N(\mathbf{A}) = \{\mathbf{x} : \mathbf{Ax} = \mathbf{0}, \mathbf{x} \in \mathbf{R}^n\} \subset \mathbf{R}^n$, $\dim N(\mathbf{A}) = n - r$
- Row space of **A** = range of $\mathbf{A}^T$ = $R(\mathbf{A}^T) = \text{span}\{\tilde{\mathbf{a}}_i\}_{i=1}^{m} \subset \mathbf{R}^n$, $\dim R(\mathbf{A}^T) = r$
- Left nullspace of **A** = nullspace of $\mathbf{A}^T$ = $N(\mathbf{A}^T) = \{\mathbf{y} : \mathbf{A}^T\mathbf{y} = \mathbf{0}, \mathbf{y} \in \mathbf{R}^m\} \subset \mathbf{R}^m$, $\dim N(\mathbf{A}^T) = m - r$

• **$\mathbf{A} \in \mathbf{R}^{m \times n}$ and echelon form U (PA = LU)**
- $R(\mathbf{A}^T) = R(\mathbf{U}^T) \subset \mathbf{R}^n$
  (a) $\dim R(\mathbf{A}^T) = \dim R(\mathbf{U}^T)$ = number of pivots = r = row rank = column rank
  (b) Basis of $R(\mathbf{A}^T) = R(\mathbf{U}^T)$ = r nonzero rows of **U**
- $N(\mathbf{A}) = N(\mathbf{U}) \subset \mathbf{R}^n$ ($\mathbf{Ax} = \mathbf{0} \Leftrightarrow \mathbf{Ux} = \mathbf{0}$)
  (a) $\dim N(\mathbf{A}) = \dim N(\mathbf{U})$ = $n - r$ = number of free variables = nullity of **A**
  (b) $\ker(\mathbf{A}) = N(\mathbf{A}) = N(\mathbf{U}) \subset \mathbf{R}^n$: kernel of **A**
  (c) For each free variable, we can generate one basis vector
- $R(\mathbf{A}) \neq R(\mathbf{U}), R(\mathbf{A}) \subset \mathbf{R}^m$
  (a) $R(\mathbf{U}) = \text{span}\{\mathbf{u}_i\}_{i=1}^{n} \subset \mathbf{R}^m$
  (b) $\dim R(\mathbf{A}) = \dim R(\mathbf{U})$ = number of pivots = r = column rank = row rank

• **Dimensions**
- $\dim R(\mathbf{A}) = \dim R(\mathbf{A}^T)$ = number of pivots = rank(**A**)
- Number of basic variables (r) + number of free variables ($n - r$) = number of columns (n) $\Leftrightarrow$ $\dim R(\mathbf{A}) + \dim N(\mathbf{A}) = n$
- $r + (m - r) = m \Leftrightarrow \dim R(\mathbf{A}^T) + \dim N(\mathbf{A}^T) = m$

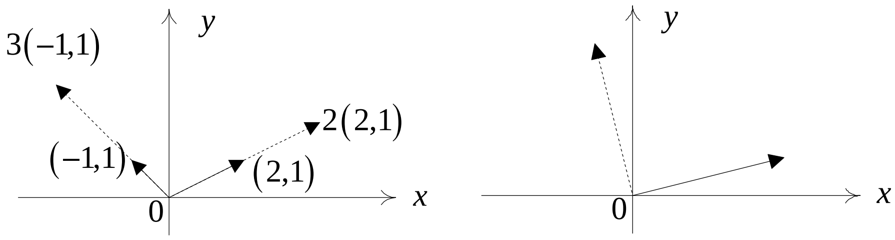
Linear Transformation, y = Ax

• Linear transformation, $\mathbf{y} = \mathbf{A}\mathbf{x}$, $\mathbf{A} \in \mathbf{R}^{m \times n}$, $\mathbf{x} \in \mathbf{R}^{n}$, $\mathbf{y} \in \mathbf{R}^{m}$

- Linearity: $\mathbf{A}(c\mathbf{x} + d\mathbf{y}) = c(\mathbf{A}\mathbf{x}) + d(\mathbf{A}\mathbf{y})$
- Examples

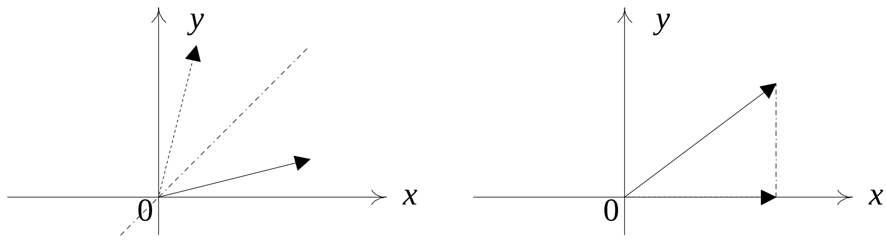
  (a) Stretching: $\mathbf{A} = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} = c\mathbf{I}$

  (b) Rotation: $\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$



  (c) Reflection: $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

  (d) Projection: $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

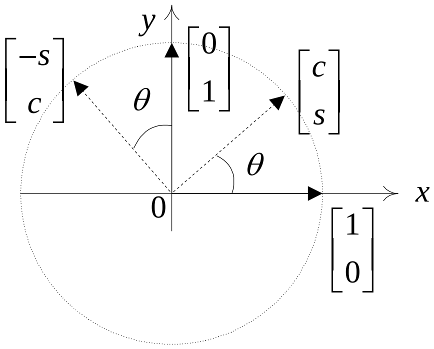


- Linear transformation matrix **A** depends on basis

• Basis and linear transformation

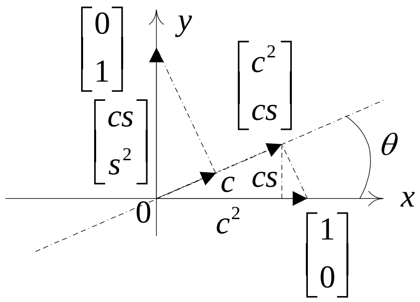
- $\mathbf{v} = \sum_{i=1}^{n} c_i \mathbf{x}_i$ with a basis $\{\mathbf{x}_i\}_{i=1}^{n}$, then $\mathbf{A}\mathbf{v} = \sum_{i=1}^{n} c_i \mathbf{A}\mathbf{x}_i$
- $\mathbf{A}: V \to W$ with $V = \text{span}\{\mathbf{x}_i\}_{i=1}^{n}$ and $W = \text{span}\{\mathbf{y}_i\}_{i=1}^{m}$ $\Rightarrow$ $\mathbf{A}\mathbf{x}_i = \sum_{j=1}^{m} a_{ji}\mathbf{y}_j$
- $\mathbf{A}: V \to W$ and $\mathbf{B}: U \to V$ $\Rightarrow$ $\mathbf{AB}: U \to W$ (i.e, $\mathbf{z} = \mathbf{A}\mathbf{y} = \mathbf{ABx}$)

• Rotation



- $\mathbf{Q}_\theta = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$
- $\mathbf{Q}_\theta \mathbf{Q}_{-\theta} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} c & s \\ -s & c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- $\mathbf{Q}_\theta^2 = \mathbf{Q}_\theta \mathbf{Q}_\theta = \mathbf{Q}_{2\theta}$ and $\mathbf{Q}_\theta \mathbf{Q}_\varphi = \mathbf{Q}_{\theta+\varphi}$

• Projection

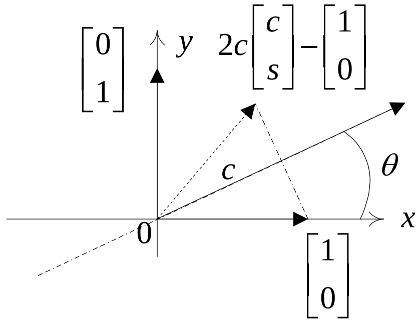


- $\mathbf{P}_\theta = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}$

- $\mathbf{P}_\theta^2 = \mathbf{P}_\theta \mathbf{P}_\theta = \mathbf{P}_\theta$: idempotent

- $\mathbf{P}_\theta^T = \mathbf{P}_\theta$: symmetric

• Reflection



- $\mathbf{H}_\theta = \begin{bmatrix} 2c^2-1 & 2cs \\ 2cs & 2s^2-1 \end{bmatrix}$

- $\mathbf{H}_\theta = 2\mathbf{P}_\theta - \mathbf{I}$ or $\mathbf{H}_\theta + \mathbf{I} = 2\mathbf{P}_\theta$

- $\mathbf{H}_\theta^2 = \mathbf{H}_\theta \mathbf{H}_\theta = \left(2\mathbf{P}_\theta - \mathbf{I}\right)^2 = 4\mathbf{P}_\theta^2 - 4\mathbf{P}_\theta^2 + \mathbf{I} = \mathbf{I} \;\Rightarrow\; \mathbf{H}_\theta^{-1} = \mathbf{H}_\theta$